A Proof of Radford's Biproduct Theorem by Using Braided Diagrams

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Abstract. We give a proof of the Radford's Biproduct Theorem in S. Montgomery's book [Hopf Algebras and Their Actions on Rings, CBMS 82, AMS,1993.] by using Majid's braided diagrams method and Yu. Bespalov and V. Lyubashenko's "t-angles.sty" package.

1 Introduction

There is an important construction in the theory of quantum groups and Hopf algebras, which is called biproduct by Radford in [3] and bosonization by Majid in [4]. Through this construction, one can get an ordinary Hopf algebra $B\star H$ from a braided Hopf algebras B, see Theorem 1.1 below. Now it is an important method of giving classification of pointed Hopf algebras through braided Hopf algebras, see [1].

Theorem 1.1. *Let H be a bialgebra, and B is an algebra in ${}_H\mathcal{M}$ and a coalgebra in ${}^H\mathcal{M}$, Then $B\star H$ becomes a bialgebra $\iff$ B is a coalgebra in ${}_H\mathcal{M}$, an algebra in ${}^H\mathcal{M}$, ε_B is an algebra map, $\delta 1_B = 1_B\otimes 1_B$, and the following identities hold:*

(1) $\delta_B(ab)=\sum a_1(a_{2(-1)}\cdot b_1)\otimes a_{2(0)}b_2$,

(2) $\sum h_1 b_{(-1)}\otimes h_2\cdot b_{(0)}=\sum (h_1\cdot b)_{(-1)}h_2\otimes (h_1\cdot b)_{(0)}$

If also B has an antipode S_B and H is a Hopf algebra with antipode S_H, then $B\star H$ is a Hopf algebra with antipode $S(b\star h)=\sum(1_B\star S_H(b_{-1}h))(S_B b_0\star 1_H)=\sum(S_H(a_{(-1)}h))_1\cdot S_B(a_{(0)})\otimes(S_H(a_{(-1)}h))_2$.

This theorem appeared in remark 10.6.6 in Montgomery's book [5]. After this remark, she said that "Although this formulation may be more natural, and has the advantage of using H and not H^{cop}, the computations are more tiresome because of the notation".

We use braided diagrams to overcome this difficulty. This method was used by Majid in [4, Theorem 2.4], but in that paper he assume H to be cocommutative. We find that this condition is not necessary and the same method can also be applied.

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Throughout the paper we freely use the notations and conventions of [3, 5] with slightly differences. In particular all vector spaces will be over a field k. For a coalgebra C with comultiplication $\Delta : C \to C \otimes C$, we write $\Delta(c) = \sum c_1 \otimes c_2$.

2 Preliminaries

Let H be a bialgebra over field k. A left H-module is a k-space M with a k-linear map $\alpha : H \otimes M \to M : \alpha(h \otimes m) = h \cdot m$ such that $(gh) \cdot m = h \cdot (g \cdot m)$. A left H-comodule is a k-space M with a k-linear map $\rho : M \to H \otimes M : \rho(m) = \sum m_{(-1)} \otimes m_{(0)}$ such that

$$\sum m_{(-1)} \otimes m_{(0)(-1)} \otimes m_{(0)(0)} = \sum m_{(-1)1} \otimes m_{(-1)2} \otimes m_{(0)}. \tag{1}$$

Here we review some basic facts about algebras and coalgebras in the category ${}_H\mathcal{M}$ of left H-modules and in the category ${}^H\mathcal{M}$ of left H-comodules. First of all recall that if M and N are left H-modules, then the left $H \otimes H$-modules $M \otimes N$ is also a left H-modules by pull-back along Δ, i.e., $h \cdot (m \otimes n) = \sum h_1 \cdot m \otimes h_2 \cdot n$.

An algebra B in ${}_H\mathcal{M}$ is an left H-module algebra, that is, B is a left H-module and also a k-algebra (B, m, η) such that m and η are module maps, or

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b) \tag{2}$$

An coalgebra B in ${}_H\mathcal{M}$ is an H-module coalgebra, that is, B is a left H-module and also a k-coalgebra (B, Δ, ε) such that Δ and ε are module maps, or

$$\Delta(h \cdot b) = \sum (h_1 \cdot b_1) \otimes (h_2 \cdot b_2) \tag{3}$$

An algebra B in ${}^H\mathcal{M}$ is an H-comodule algebra, that is, B is a left H-comodule and also a k-algebra (B, m, η) such that m and η are module maps, or

$$\rho(ab) = \sum a_{(-1)} b_{(-1)} \otimes a_{(0)} b_{(0)} \tag{4}$$

An coalgebra B in ${}^H\mathcal{M}$ is an H-comodule coalgebra, that is, B is a left H-comodule and also a k-coalgebra (B, Δ, ε) such that Δ and ε are comodule maps, or

$$\sum b_{(-1)} \otimes b_{(0)1} \otimes b_{(0)2} = \sum b_{1(-1)} b_{2(-1)} \otimes b_{1(0)} \otimes b_{2(0)} \tag{5}$$

3 The Proof

In this section, we give a detailed proof of Theorem 1.1. First, we review some facts about the product and coproduct in $B \star H$.

For B be an left H-module algebra. Then the smash product algebra $B\#H$ is defined as follows: $B\#H = B \otimes H$ as k-spaces, with multiplication given by

$$(a\#g)(b\#h) = \sum a(h_1 \cdot b)\#h_2 g$$

Let H be a bialgebra and B a coalgebra in ${}^H\mathcal{M}$. The smash coproduct $B\sharp H$ is defined to be $B \otimes H$ as a vector space, with comultiplication given by

$$\Delta(b\sharp h) = \sum b_1 \sharp b_{2(-1)} h_1 \otimes b_{1(0)} \sharp h_2,$$

and counit $\varepsilon(b\sharp h) = \varepsilon(b)\varepsilon(h)$, for all $b \in B, h \in H$.

Lemma 3.1. *$B\sharp H$ is a coalgebra with the above comultiplication.*

Proof. We check that Δ is coassociative. Now for $b \in B, h \in H$,

$$\begin{aligned}
& (\Delta \otimes \mathrm{id})\Delta(b\sharp h)) \\
= & \sum b_{11} \sharp b_{12(-1)} (b_{2(-1)} h_1)_1 \otimes b_{12(0)} \sharp (b_{2(-1)} h_1)_2 \otimes b_{2(0)} \sharp h_2 \\
= & \sum b_1 \sharp b_{2(-1)} b_{3(-2)} h_1 \otimes b_{2(0)} \sharp b_{3(-1)} h_2 \otimes b_{3(0)} \sharp h_3 \\
& \text{by equation (1) applied to } b_3 \\
= & \sum b_1 \sharp b_{2(-1)} b_{3(-1)} h_1 \otimes b_{2(0)} \sharp b_{3(0)(-1)} h_2 \otimes b_{3(0)(0)} \sharp h_3 \\
& \text{by equation (5) applied to } b_2 \\
= & \sum b_1 \sharp b_{2(-1)} h_1 \otimes b_{2(0)} \sharp b_{2(0)2(-1)} h_2 \otimes b_{2(0)2(0)} \sharp h_3 \\
= & (\mathrm{id} \otimes \Delta)\Delta(b\sharp h))
\end{aligned}$$

Proof of Theorem 1.1 We will show that $B \star H$ is a bialgebra. First, $B \star H$ is an algebra by smash product, and it is a coalgebra by Lemma 3.1. We now check that

$$\begin{aligned}
& \Delta((a \star g)(b \star h)) = \Delta(a(g_1 b) \star g_2 h) \\
= & \sum (a(g_1 b))_1 \star (a(g_1 b))_{2(-1)} (g_1 h)_1 \otimes (a(g_1 b))_{2(0)} \star (g_2 h)_2 \\
& \text{by condition (1) in Theorem 1.1 applied to } a(g_1 b) \\
= & \sum a_1 (a_{2(-1)} (g_1 b)_1) \star (a_{2(-1)} (g_1 b)_2)_{(-1)} (g_2 h)_1 \otimes (a_{2(-1)} (g_1 b)_2)_{(0)} \star g_{22} h_2 \\
& \text{by equation (3) applied to } g_1 \cdot b \text{ and equation (4) applied to } a_{2(0)} (g_1 b)_2 \\
= & \sum a_1 (a_{2(-1)} (g_{11} b_1)) \star (a_{2(-1)} (g_{12} b_2))_{(-1)} (g_{21} h_1) \otimes (a_{2(-1)} (g_{12} b_2))_{(0)} \star g_{22} h_2 \\
= & \sum a_1 (a_{2(-1)} (g_1 b_1)) \star a_{2(0)(-1)} (g_2 b_2) g_3 h_1 \otimes a_{2(0)(0)} (g_2 h_2)_{(0)} \star g_4 h_2 \\
= & \sum a_1 (a_{2(-1)} (g_1 b_1)) \star a_{2(0)(-1)} g_2 b_{2(-1)} h_1 \otimes a_{2(0)(0)} (g_3 h_{2(0)}) \star g_4 h_2 \\
& \text{by condition (2) in Theorem 1.1 applied to } g_2 \otimes b_2 \\
= & \sum a_1 ((a_{2(-1)1} g_1 b_1) \star a_{2(-1)2} g_2 b_{2(-1)} h_1 \otimes a_{2(0)} (g_3 h_{2(0)}) \star g_4 h_2
\end{aligned}$$

$$
\begin{aligned}
&= \sum a_1((a_{2(-1)}g)_1 b_1) \star (a_{2(-1)}g_1)_2 b_{2(-1)} h_1 \otimes a_{2(0)}(g_3 h_{2(0)}) \star g_4 h_2 \\
&= \Delta(a \star g)\Delta(b \star h)
\end{aligned}
$$

The fact that the given S make $B \star H$ into a Hopf algebra was stated without proof in [3, 5]. We give the detailed proof below.

$$
\begin{aligned}
& (S * \mathrm{id})\Delta(b \star h) = S(b \star h)_1 (b \star h)_2 \\
= \; & (S_H(b_{1(-1)} b_{2(-1)} h_1)_1 \cdot S_B(b_{1(0)}))(S_H(b_{1(-1)} b_{2(-1)} h_1)_2 \cdot b_{2(0)}) \otimes S_H(b_{1(-1)} b_{2(-1)} h_1)_3 h_2 \\
= \; & (S_H(b_{(-1)} h_1)_1 \cdot S_B(b_{(0)1})(S_H(b_{(-1)} h_1)_2 \cdot b_{(0)2}) \otimes S_H(b_{1(-1)} h_1)_3 h_2 \\
& \text{by } B \text{ is an } H\text{-comodule coalgebra} \\
= \; & (S_H(b_{(-1)} h_1)_1 \cdot (S_B(b_{(0)1}) b_{(0)2}) \otimes S_H(b_{(-1)} h_1)_2 h_2 \\
& \text{by } B \text{ is an } H\text{-module algebra applied to } S_H(b_{(-1)} h_1)_1 \otimes S_B(b_{(0)1}) \otimes b_{(0)(-1)} \\
= \; & \varepsilon_B(b_{(0)})(S_H(b_{(-1)} h_1)_1 \cdot 1_H) \otimes S_H(b_{(-1)} h_1)_2 h_2 \\
& \text{by antipode of } B \text{ applied to } b_{(0)} \\
= \; & \varepsilon_B(b)\varepsilon_B((S_H(1_H h_1))_1) 1_B \otimes S_H(1_H h_1)_2 h_2 \\
= \; & \varepsilon_B(b)\varepsilon_B(S_H(1_H h_1)_1 1_H \otimes S_H(1_H h_1)_2 h_2 \\
= \; & \varepsilon_B(b) 1_B \otimes S(h_1) h_2 \\
= \; & \varepsilon_B(b)\varepsilon(h) 1_B \otimes 1_H
\end{aligned}
$$

$$
\begin{aligned}
& (\mathrm{id} * S)\Delta(b \star h) = (b \star h)_1 S(b \star h)_2 \\
= \; & b_1((b_{2(-1)} h_1)_1 \cdot (S_H(a_{2(0)(-1)} h_2)_1 \cdot S_B(a_{2(0)(0)})) \otimes (b_{2(-1)} h_1)_2 S_H(a_{2(0)(-1)} h_2)_2 \\
= \; & b_1((b_{2(-1)1} h_1)_1 S_H(b_{2(-1)1} h_2)_1 \cdot S_B(b_{2(0)})) \otimes (b_{2(-1)1} h_1)_{(-1)} S_H(b_{2(-1)1} h_2)_2 \\
= \; & b_1(((b_{2(-1)1} h)_1 S_H(b_{2(-1)1} h)_2)_1 \cdot S_B(b_{2(0)})) \otimes (b_{2(-1)1} h)_1 S_H(b_{2(-1)1} h)_2)_2 \\
= \; & \varepsilon_B(b_{2(-1)} h) b_1(1_H \cdot S_B(b_{2(0)})) \otimes 1_H \\
= \; & \varepsilon_B(b_{2(-1)}) b_1(1_H \cdot S(b_{2(0)})) \otimes \varepsilon_H(h) 1_H \\
= \; & b_1 S(b_2) \otimes \varepsilon_H(h) 1_H \\
= \; & \varepsilon_B(b)\varepsilon(h) 1_B \otimes 1_H
\end{aligned}
$$

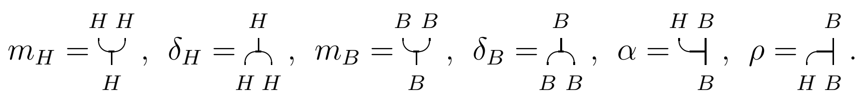
When we apply the biproduct construction to the category of quasitiangular Hopf algebras, we get the following result. It is due to Majid [4] which he called "bosonization".

Proposition 3.2. *If (H, R) be a quasitriangular bialgebra and let B be a bialgebra in ${}_H\mathcal{M}$. Then B is also a left H-comodule algebra by defining $\rho : B \to H \otimes B$ via $\rho(b) = R^{-1}(1 \otimes b)$ for all $b \in B$. Then the biproduct $B \star H$ is a bialgebra. If also H is a Hopf algebra and B is a Hopf algebra in ${}_H\mathcal{M}$, then $B \star H$ is a Hopf algebra.*

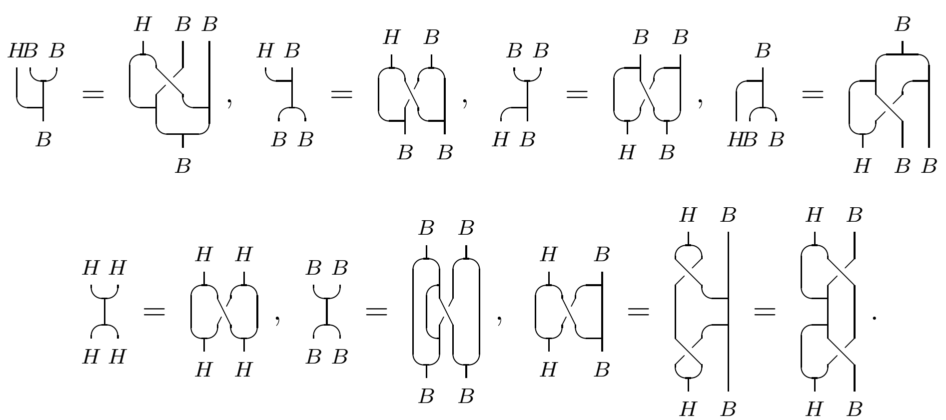
4 Braided diagrams: the method we use

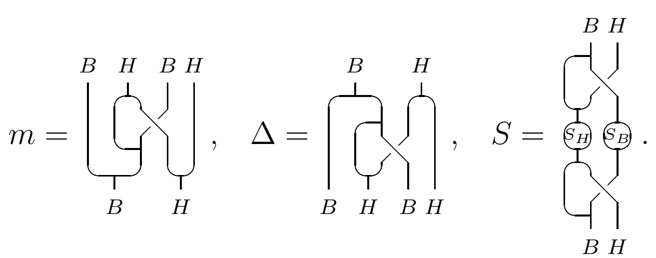
In this section, we give an introduction of the method of braided diagrams in Hopf algebras from which we get the above results. This method was introduced by D. N. Yetter in [6] where he found that the category of Yetter-Drinfeld modules form a braided monoidal category. There he call it crossed bimodules which in fact is condition (2) of Theorem 1.1. It was used by many authors such as S.Majid in [4], Y.Bespalov and B. Drabant in [2], S.C. Zhang and H.X. Chen in [7], etc. Here we use Yu. Bespalov and V. Lyubashenko's "t-angles.sty" package as in [2] to draw the diagrams.

First, all maps are written downwards from top to bottom. The maps m, Δ, α, ρ are graphically written as



Secondly, we can picture the conditions of module algebra, module coalgebra, comodule algebra, comodule coalgebra, algebra-coalgebra, Yetter-Drinfel'd modules as follows:





Finally, we give the diagrammatic proof of Lemma 3.1 and Theorem 1.1.

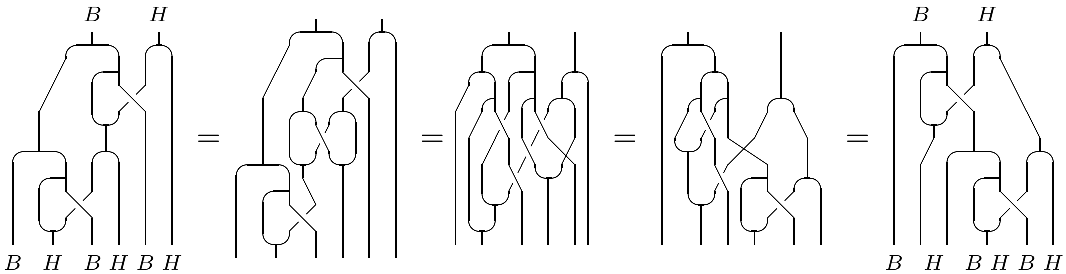


Figure 1: Proof of coalgebra structure of $B \star H$

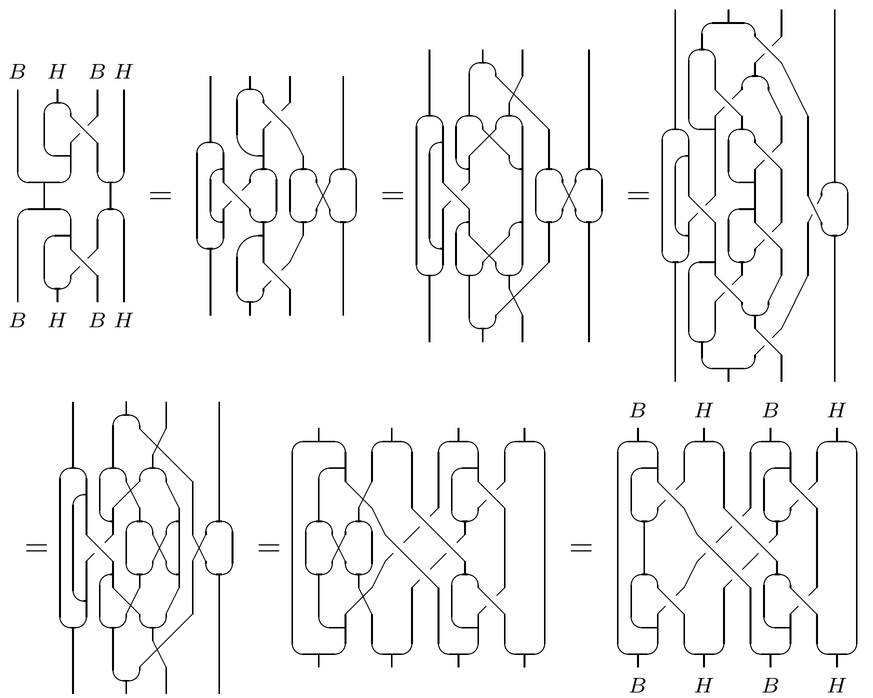
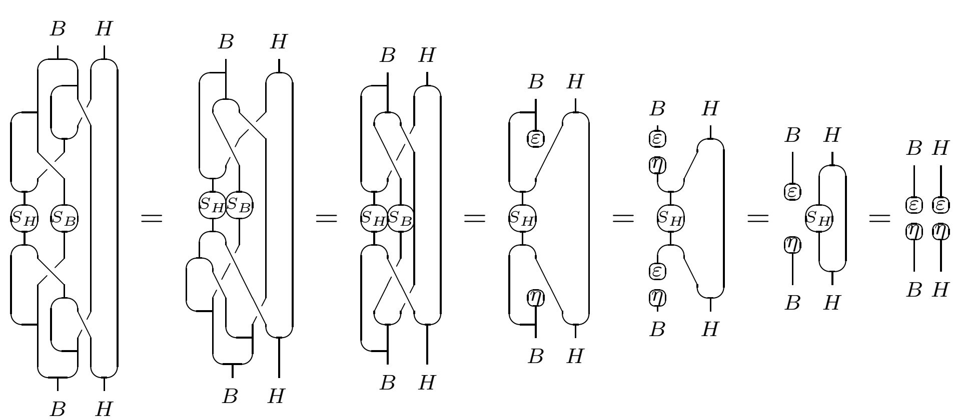


Figure 2: Proof of bialgebra structure of $B \star H$



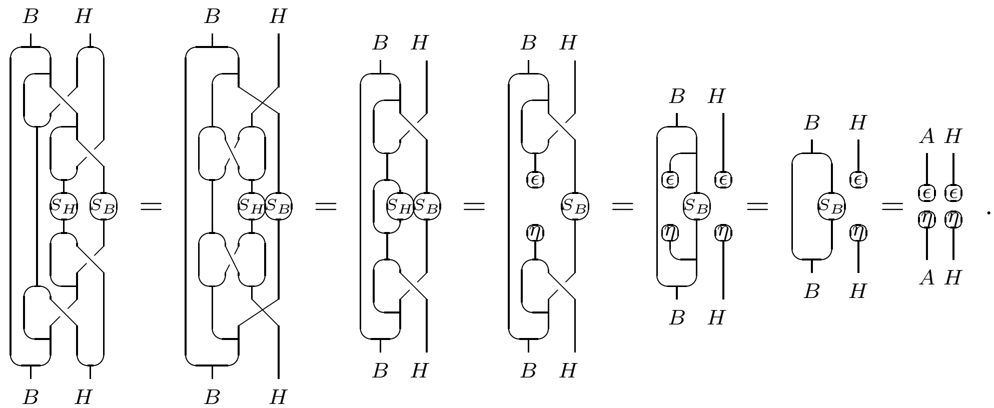


Figure 3: Proof of antipode of $B \star H$

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